July 23, 2020

Signal Processing Basics

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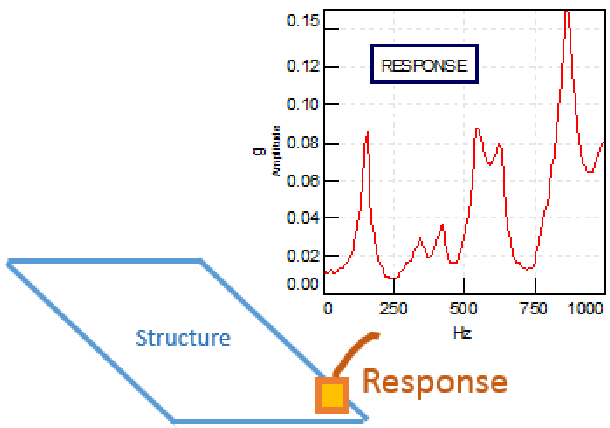
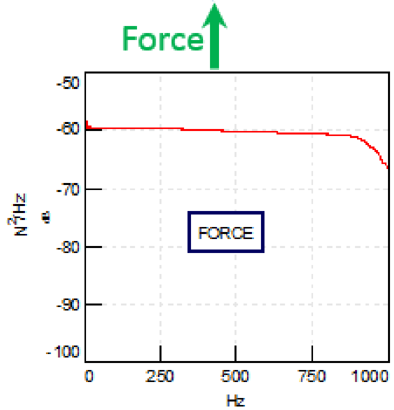
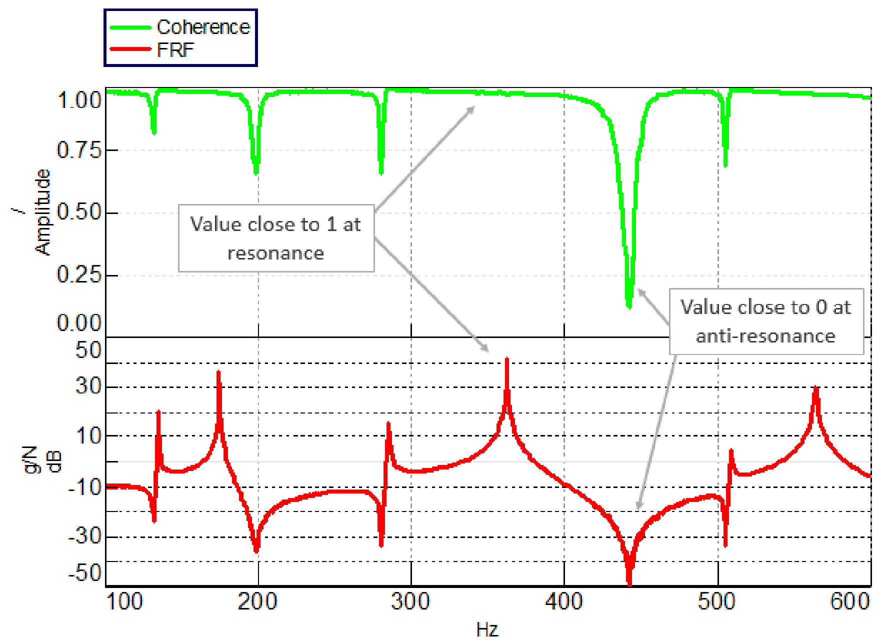
Frequency Response Function

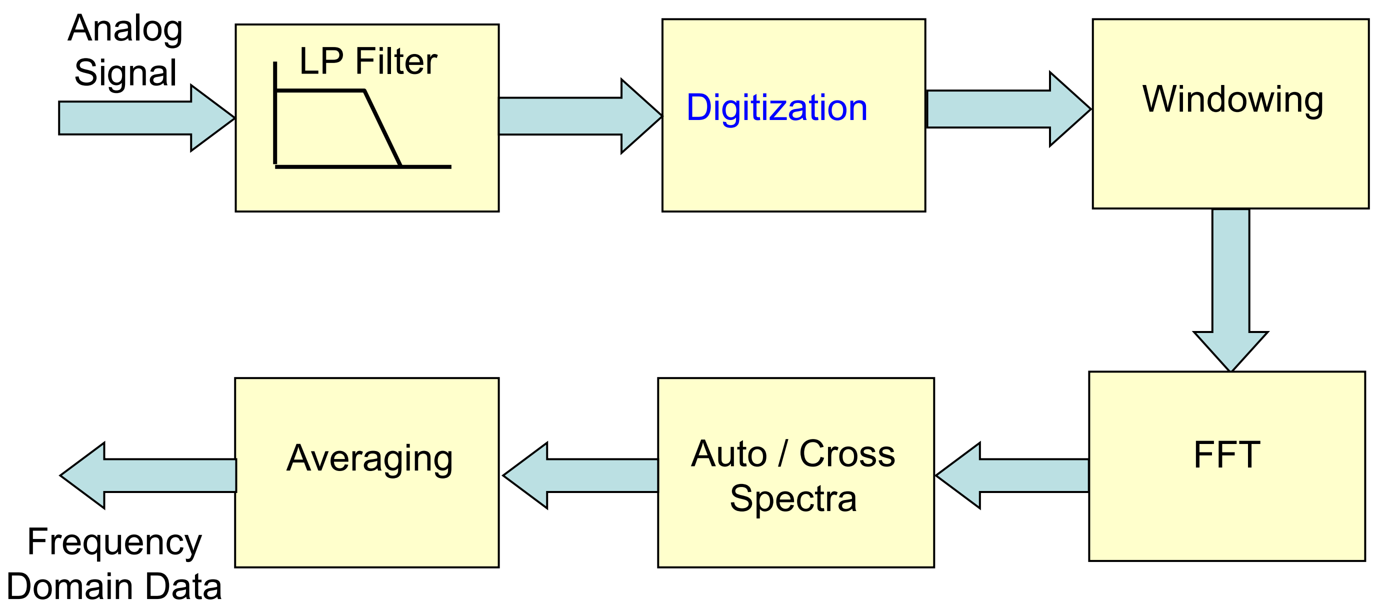






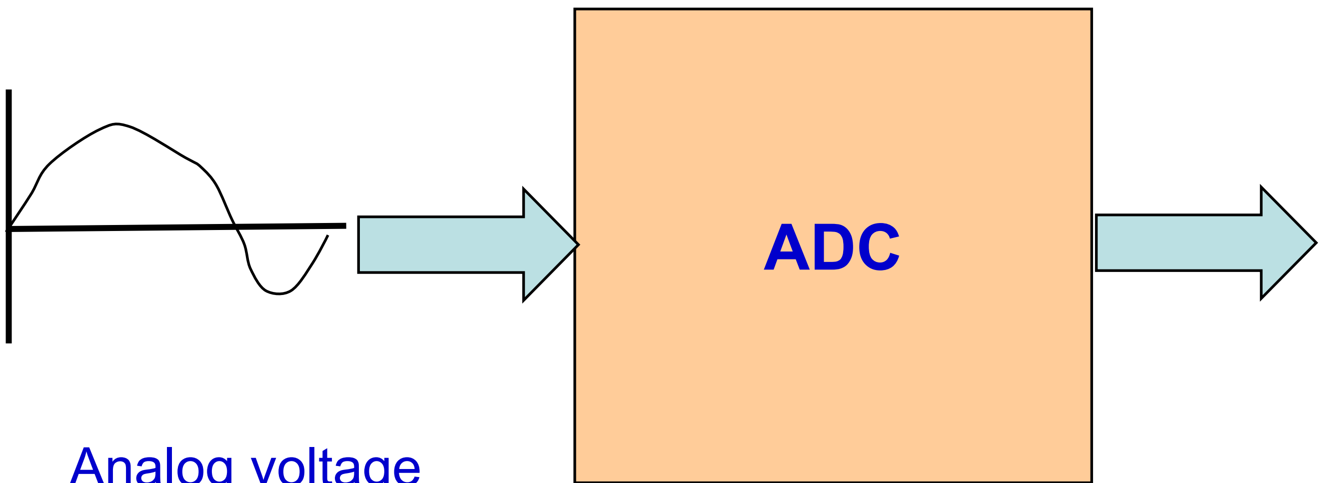
https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/What-is-a-Frequency-Response-Function-FRF/ta-p/354778

Signal Processing



Digitization Analog to Digital

Time signals are sampled using an *analog-to-digital converter* (a digital logic hardware device) to yield a digitized version of the waveform for further processing.

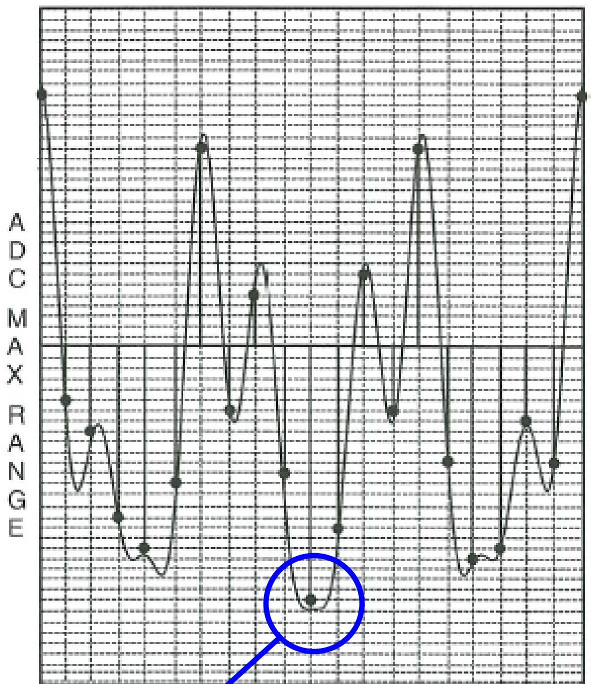


Time (s)	Voltage (V)
0.0	0.0
0.001	0.15
0.002	0.18
0.003	0.28
etc.	

Analog voltage signal from transducer

Voltage values stored in computer memory

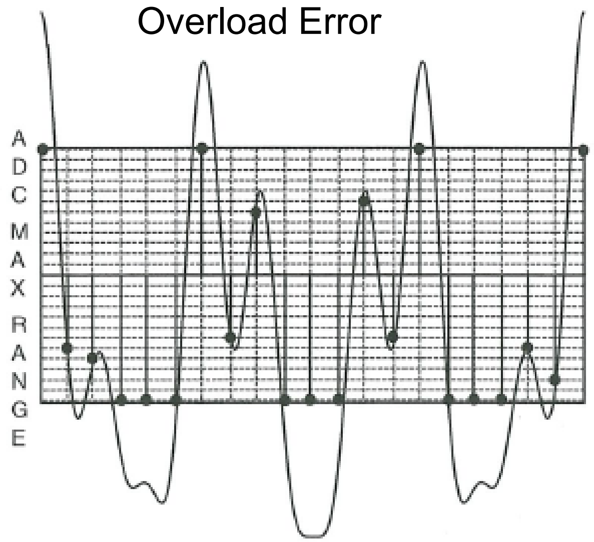
Digitization Quantization Errors



(a)

Quantization Error

Overload Error

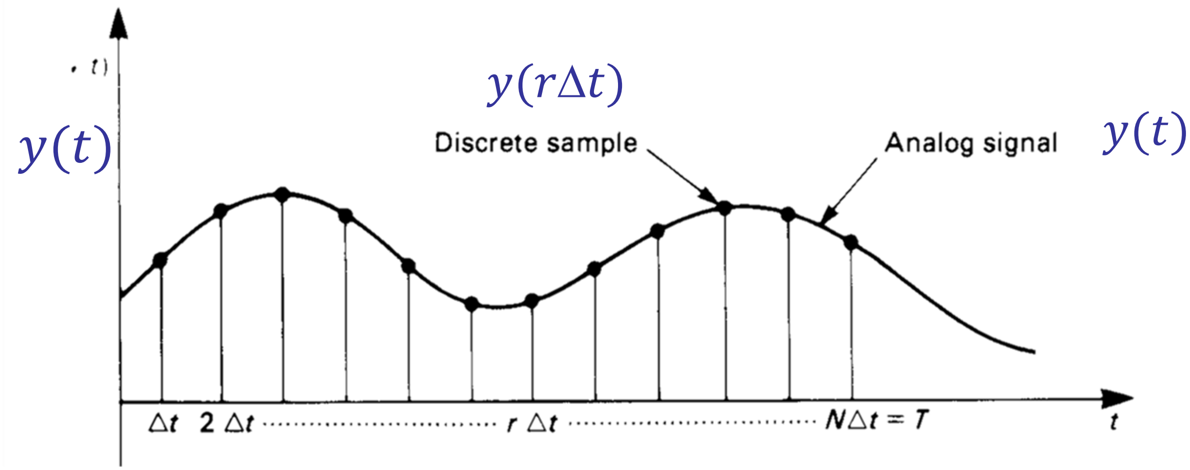


(b)

From P. Avitabile (2017).

A 24-bit ADC can capture up to 2^{24} (~16.8 million) levels. Audio CD's are 16-bit (~65.5 thousand levels).

Digitization Time Domain Terms



Δt = sample interval

f_s = sampling rate = $1/\Delta t$

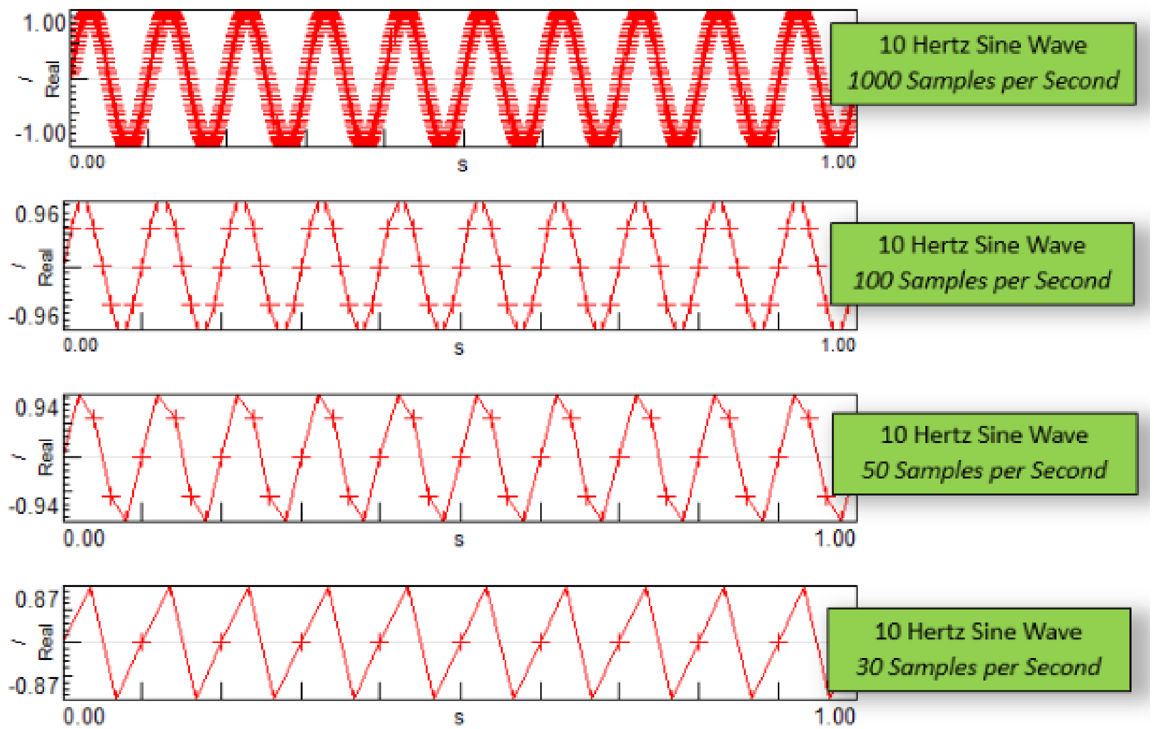
N = total number of data samples in one frame

T = total sample period or frame size

r = sample index number $(1, 2, 3, \ldots, N)$

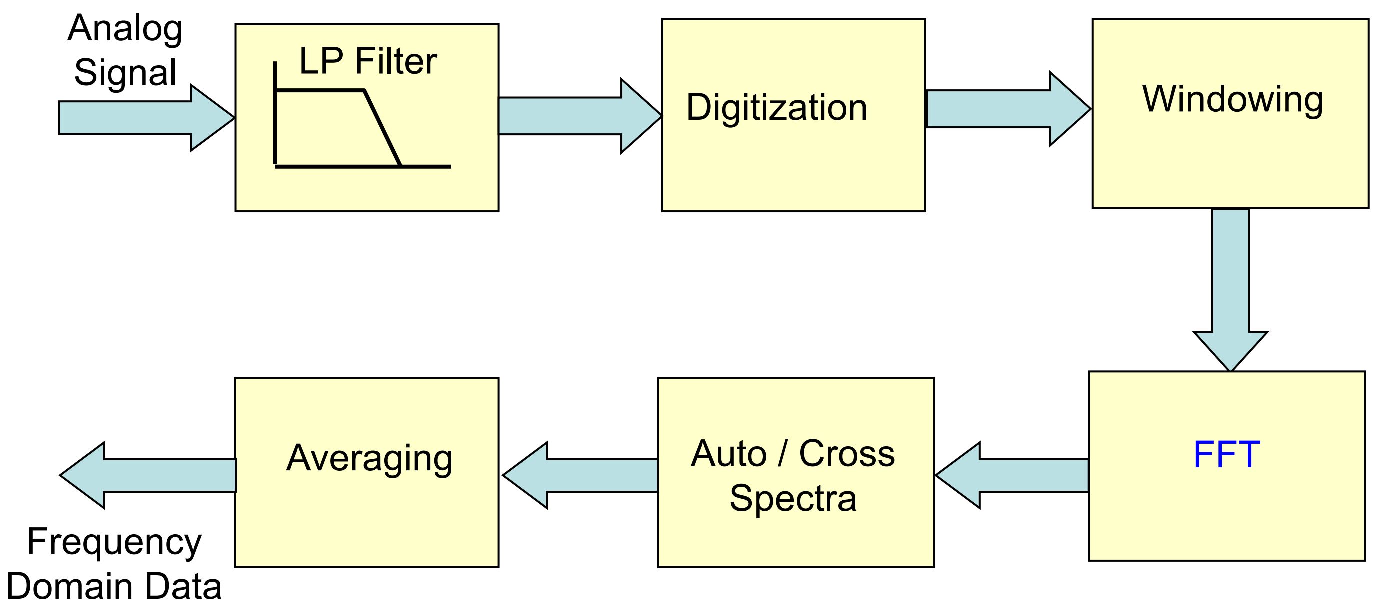
$t = r\Delta t$ time of any given sample

Digitization Sample Rate

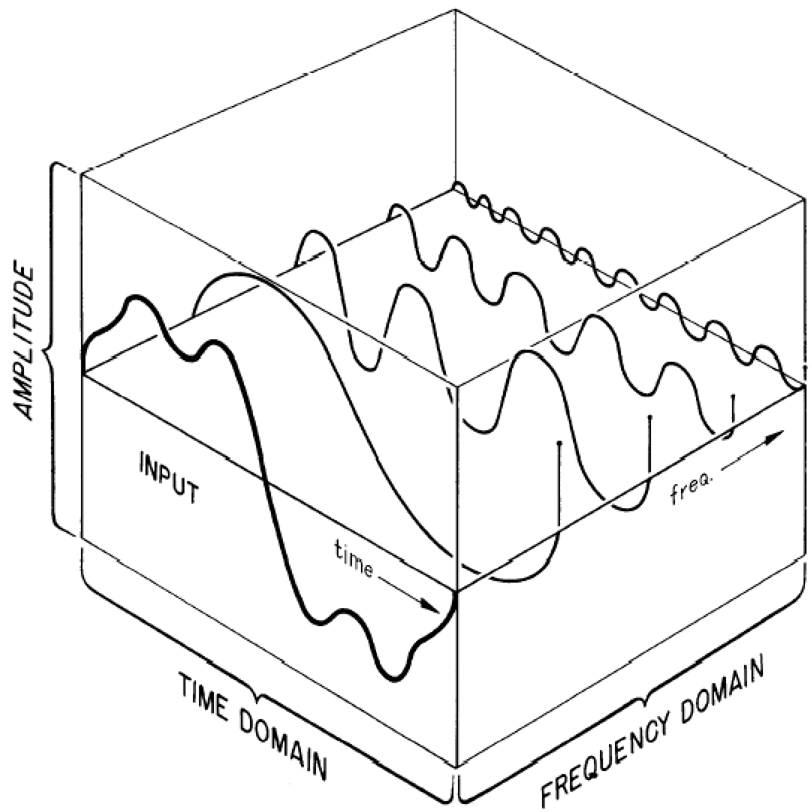


https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Digital-Signal-Processing-Sampling-Rates-Bandwidth-Spectral/ta-p/402991

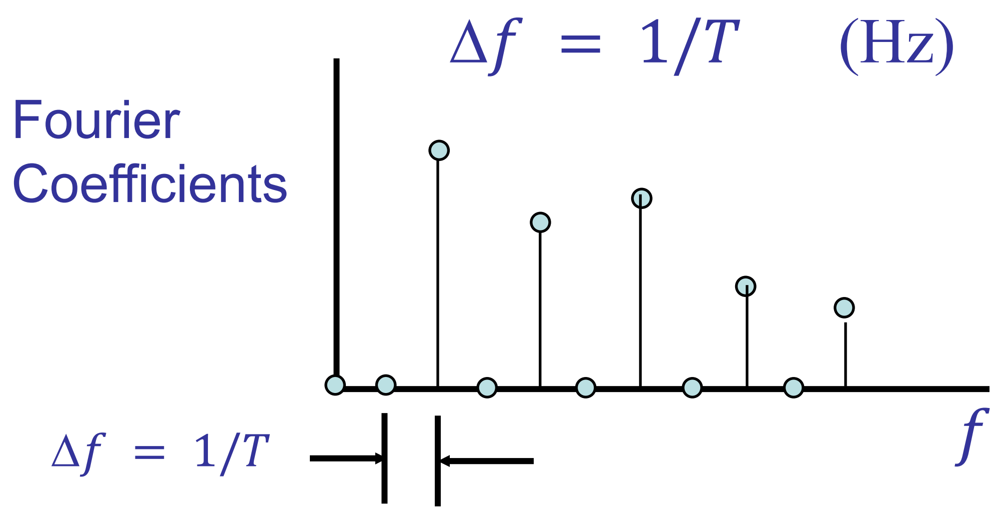
Signal Processing



Time and Frequency Domains



FFT Frequency Domain Terms



$\Delta f = 1/T$ frequency resolution

number of spectral lines (total number of frequency samples)

bandwidth is the highest frequency captured in Fourier transform

Fourier Transform

Used to determine the frequency spectrum of dynamic signals – spectrum analyzer hardware

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(n\omega t) + B_n \sin(n\omega t))$$

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (C_n \cos(n\omega t - \phi_n))$$

$$A_0 = \frac{2}{T}\int_0^T y(t)dt \qquad A_n = \frac{2}{T}\int_0^T y(t)\cos(n\omega t)\,dt \qquad B_n = \frac{2}{T}\int_0^T y(t)\sin(n\omega t)\,dt$$

$$C_n = \sqrt{A_n^2 + B_n^2} \qquad \phi_n = \tan^{-1}\left(\frac{B_n}{A_n}\right)$$

$T = 2\pi/\omega$ Is the period of the signal, ω is the *fundamental* frequency (first harmonic), 2ω is the *second harmonic*, etc.

FFT Fast Fourier Transform

Discrete (digital) form:

$$A_n = \frac{2}{N\Delta t}\sum_{r=1}^{N} y(r\Delta t)\cdot\cos\left(\frac{2\pi}{N\Delta t}\cdot n\cdot r\Delta t\right)\Delta t = \frac{2}{N}\sum_{r=1}^{N} y(r\Delta t)\cdot\cos\left(\frac{2\pi}{N}\cdot n\cdot r\right) \quad n = 1,2,\ldots,\frac{N}{2}$$

$$B_n = \frac{2}{N}\sum_{r=1}^{N} y(r\Delta t)\cdot\sin\left(\frac{2\pi}{N}\cdot n\cdot r\right) \quad n = 1,2,\ldots,\frac{N}{2}$$

The DFT is a CPU intensive algorithm. N^2 operations are required. If we split the signal into parts and then take the DFT for each part, we can reduce the number of operations. The computation will take less than 1% of the original time if a smart algorithm referred to as the FFT is used. There are many different FFT recipes but all require a power of 2 number of samples i.e., 512, 1024, 2048, etc. (Bodén, Ahlin, and Carlsson, Signal Analysis)

FFT Settings

Pick	Then	And
Δt	$f_{max} = 1/2\Delta t$	$T = N\Delta t$ $\Delta f = 1/N\Delta t$
f_{max}	$\Delta t = 1/2f_{max}$	
Δf	$T = 1/\Delta f$	$\Delta t = T/N$ $f_{max} = N\Delta f/2$
T	$\Delta f = 1/T$	

Example

$\Delta f = 5$ Hz and $N = 1024$

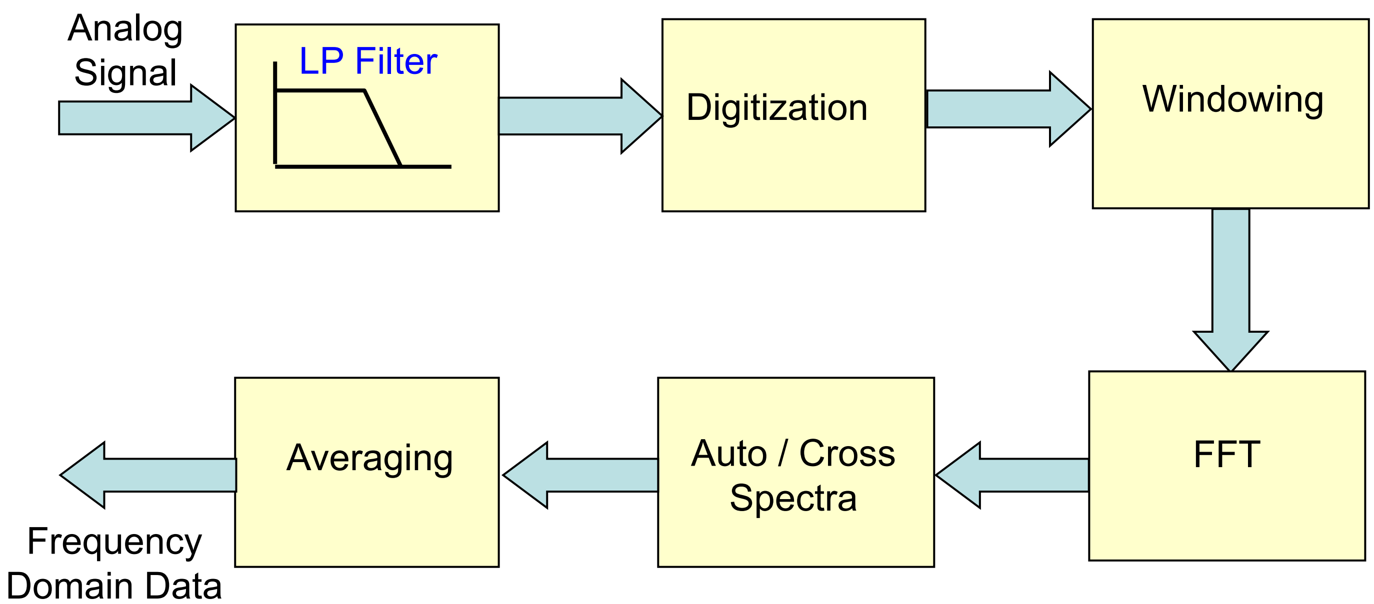
Then

$T = 1/\Delta f = 1/5$ Hz $= 0.2$ sec

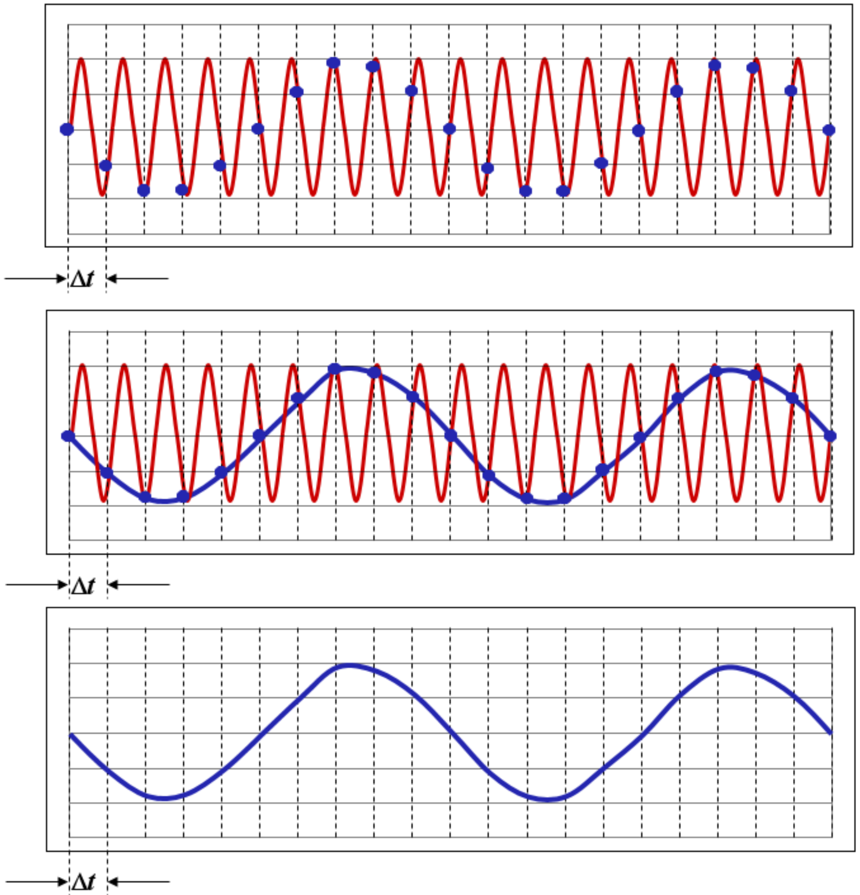
$f_s = N\Delta f = (1024)(5)$ Hz $= 5210$ Hz

$f_{max} = f_s = 5120/2$ Hz $= 2560$ Hz

Signal Processing



LP Filter The Aliasing Phenomenon

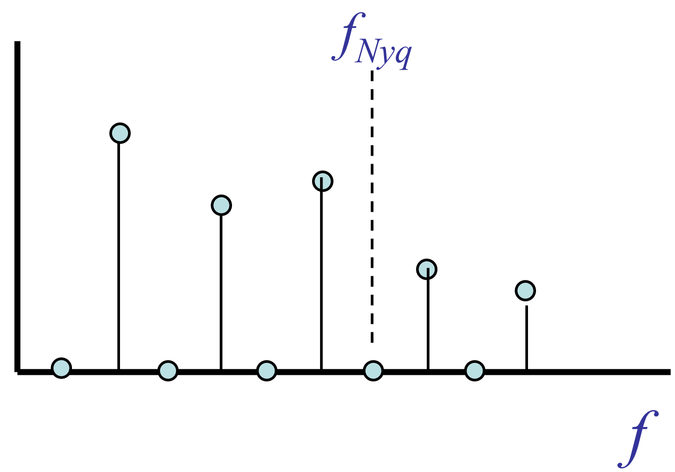


Nyquist frequency (f_{max}) is the highest frequency resolved in a DFT is:

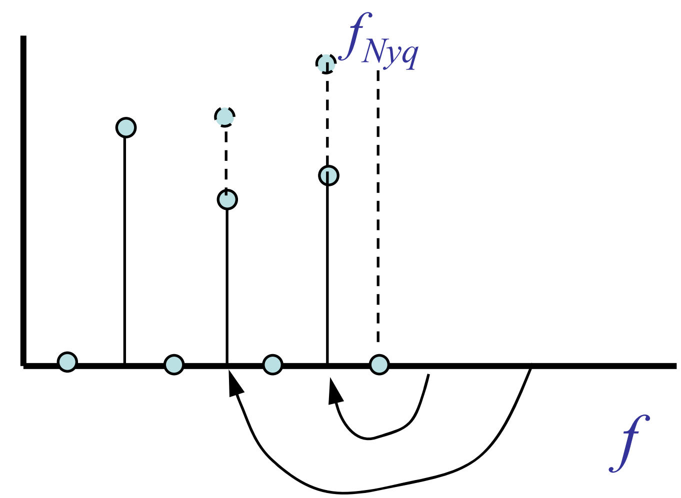
$$f_{max} = \frac{f_s}{2}$$

https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Aliasing/ta-p/367750

LP Filter Aliasing Phenomenon



True spectrum



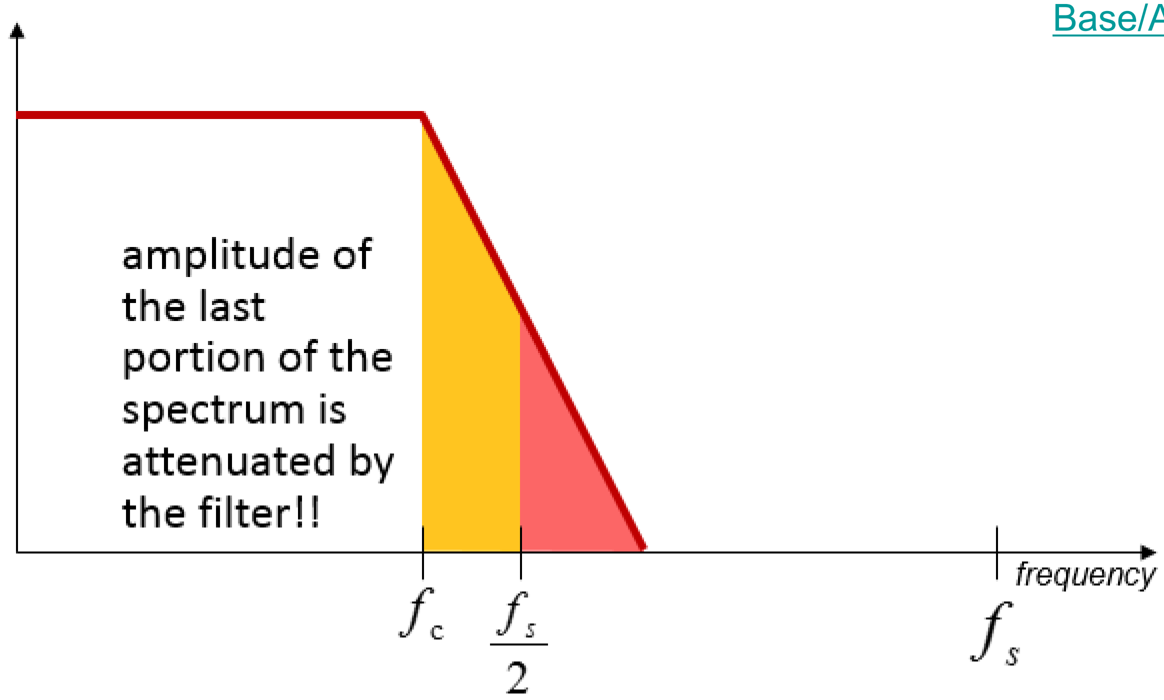
Measured spectrum

Aliasing results in harmonics above the Nyquist frequency being "folded" back onto their neighbors.

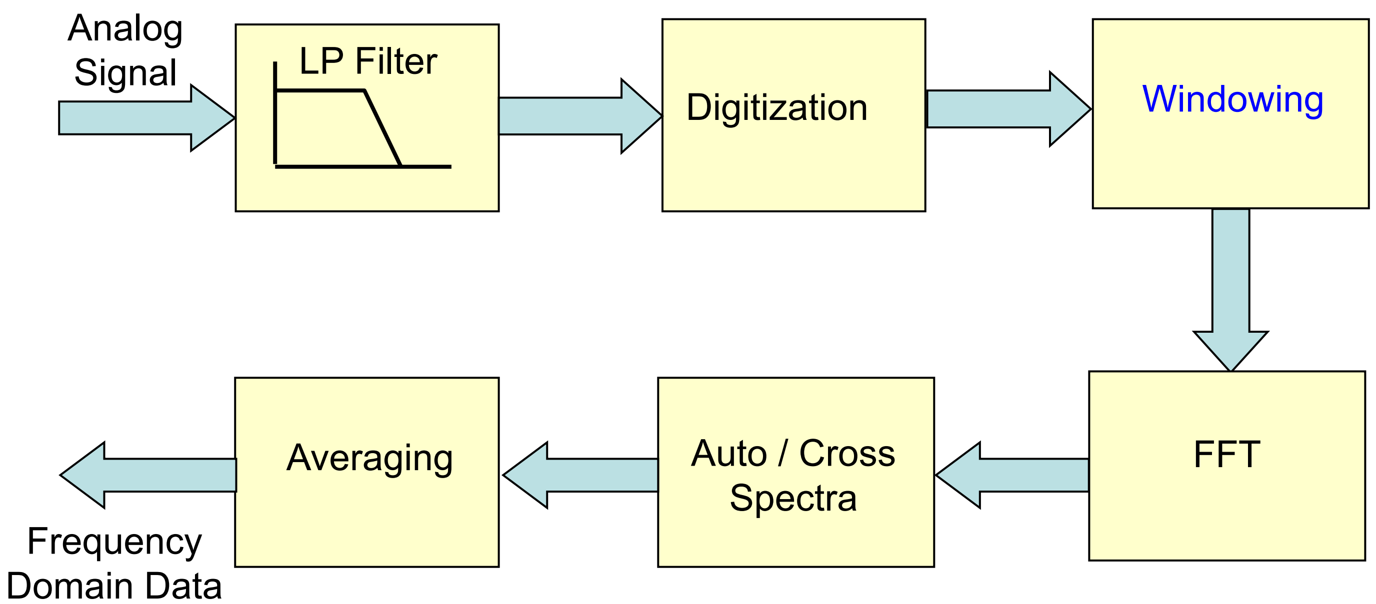
LP Filter Preventing Aliasing

Apply an analog low-pass filter

https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/Aliasing/ta-p/367750

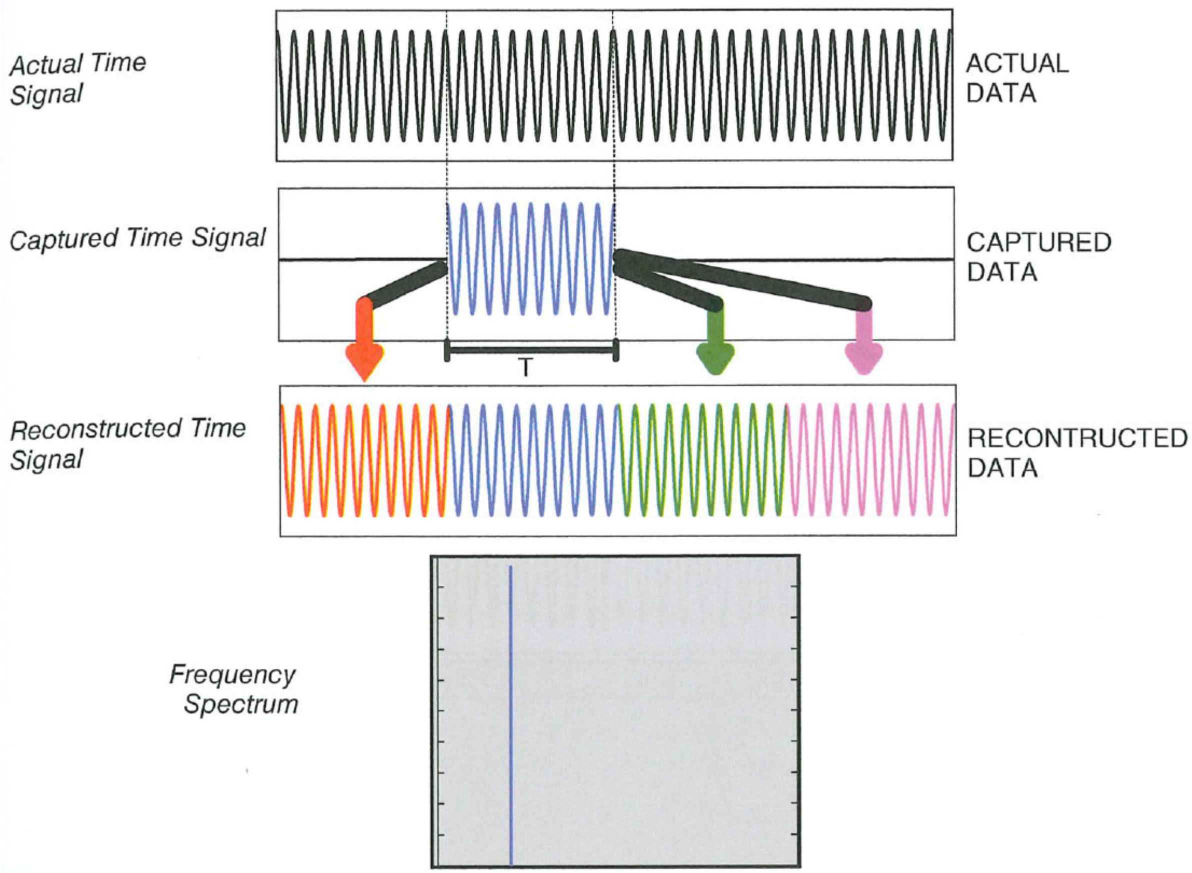


Signal Processing



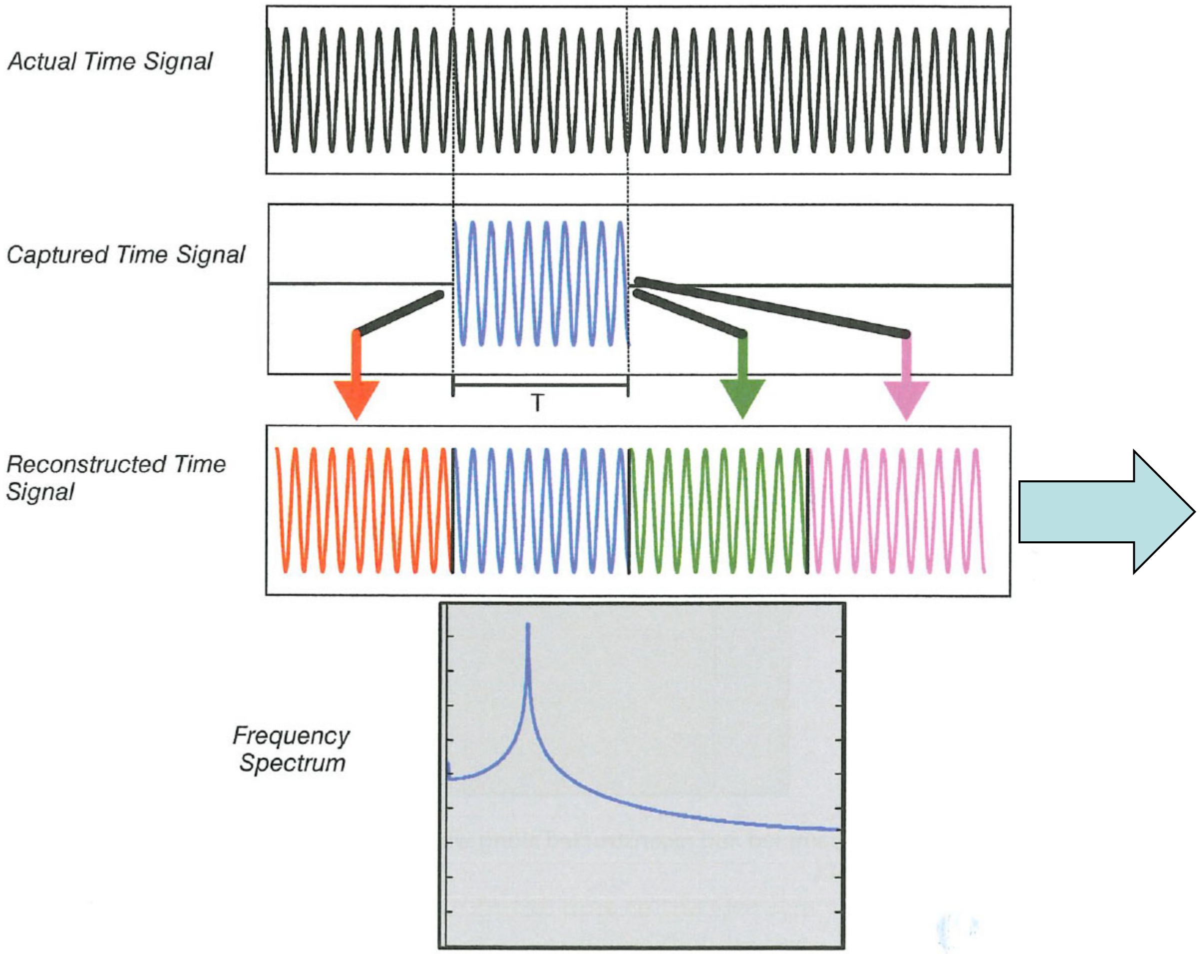
Windowing What is Leakage?

Ideally



From P. Avitabile (2017).

Windowing What is Leakage?



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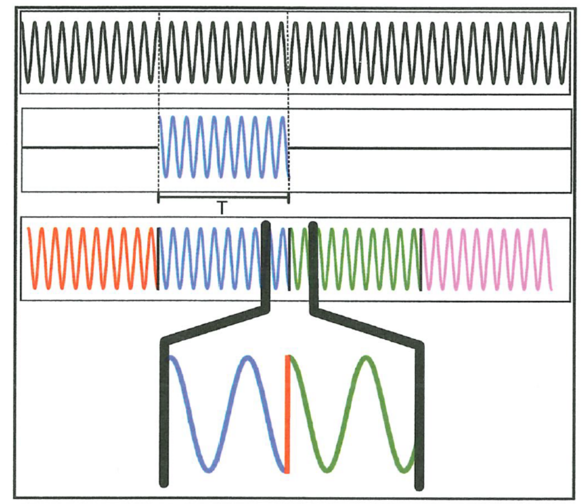
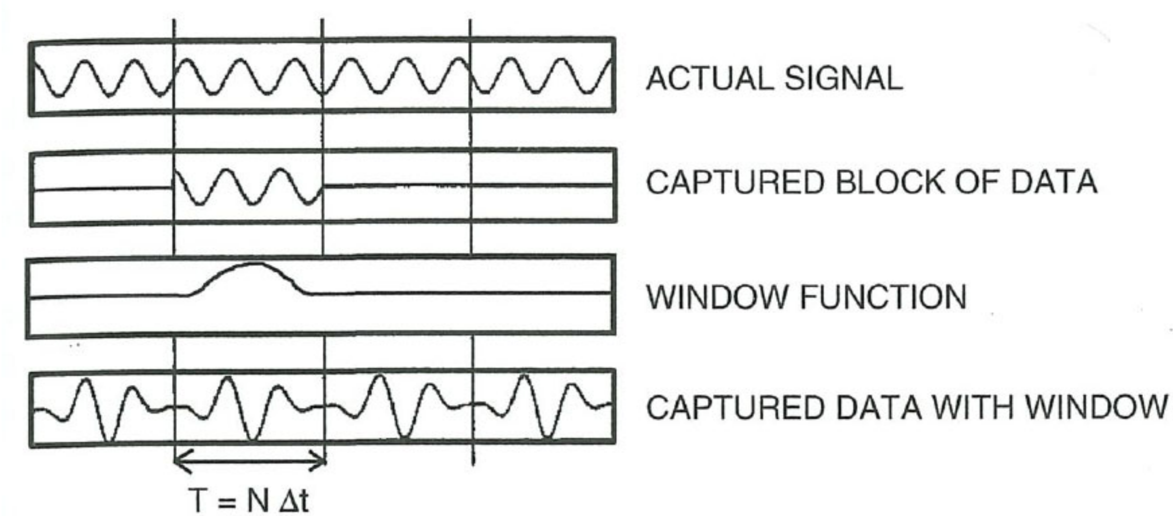


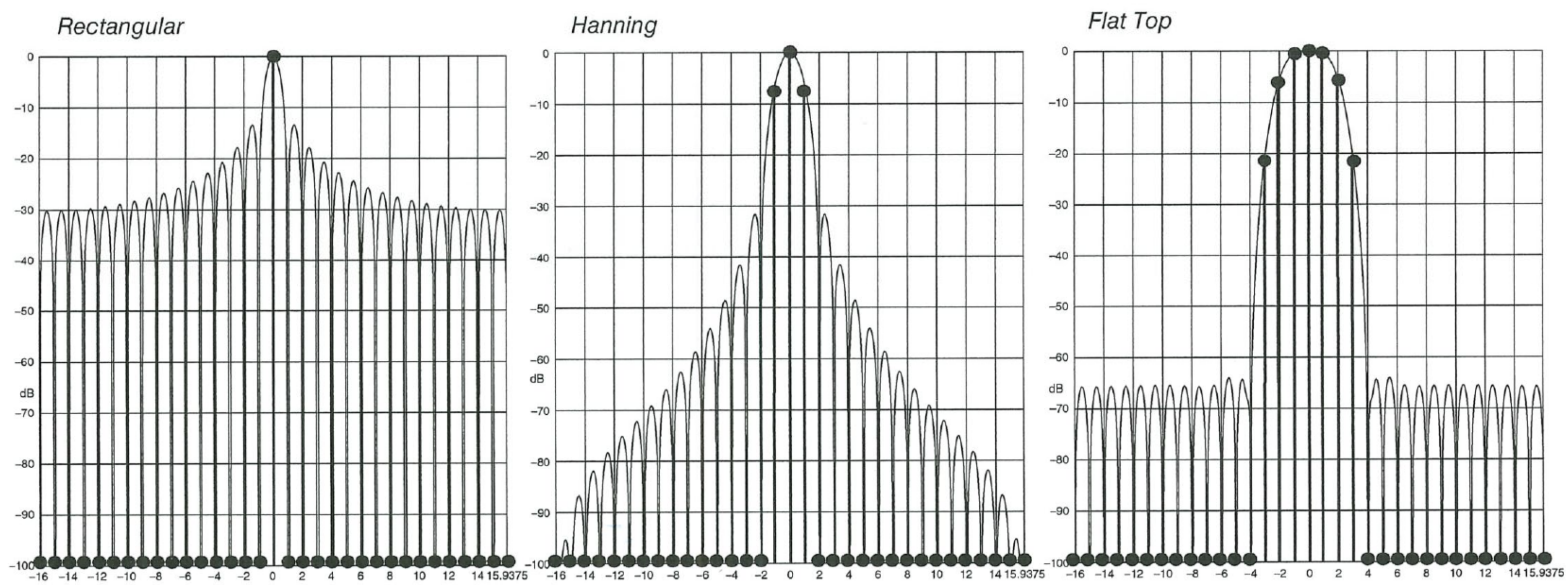
Windowing What is Leakage?



From P. Avitabile (2017).

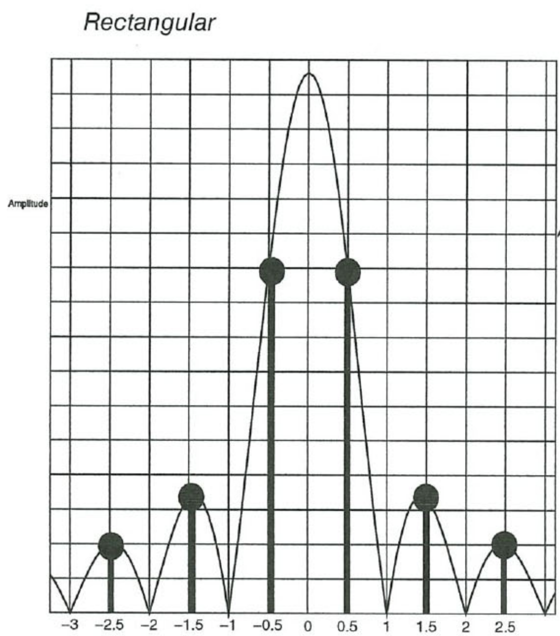
Windowing Applying a Window

Capturing an integer number of periods (minimum error).

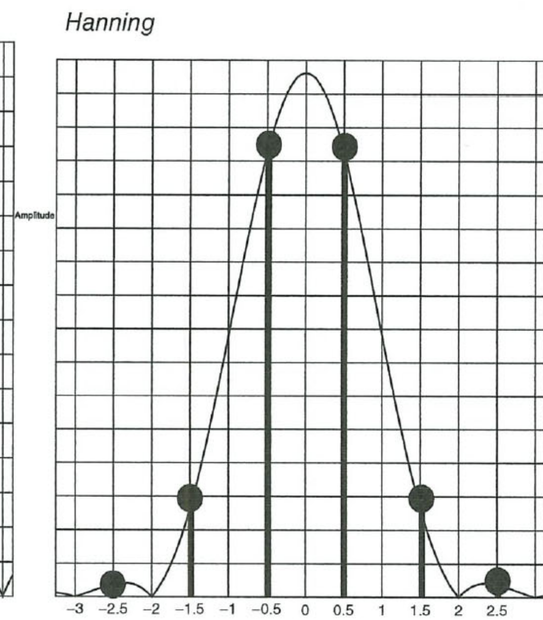


Windowing Applying a Window

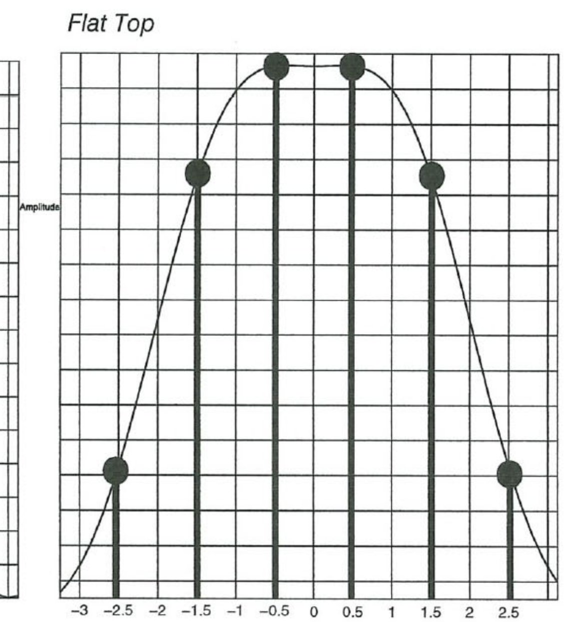
Capturing an integer number of periods plus half a period (maximum error).



3.9 dB amplitude error

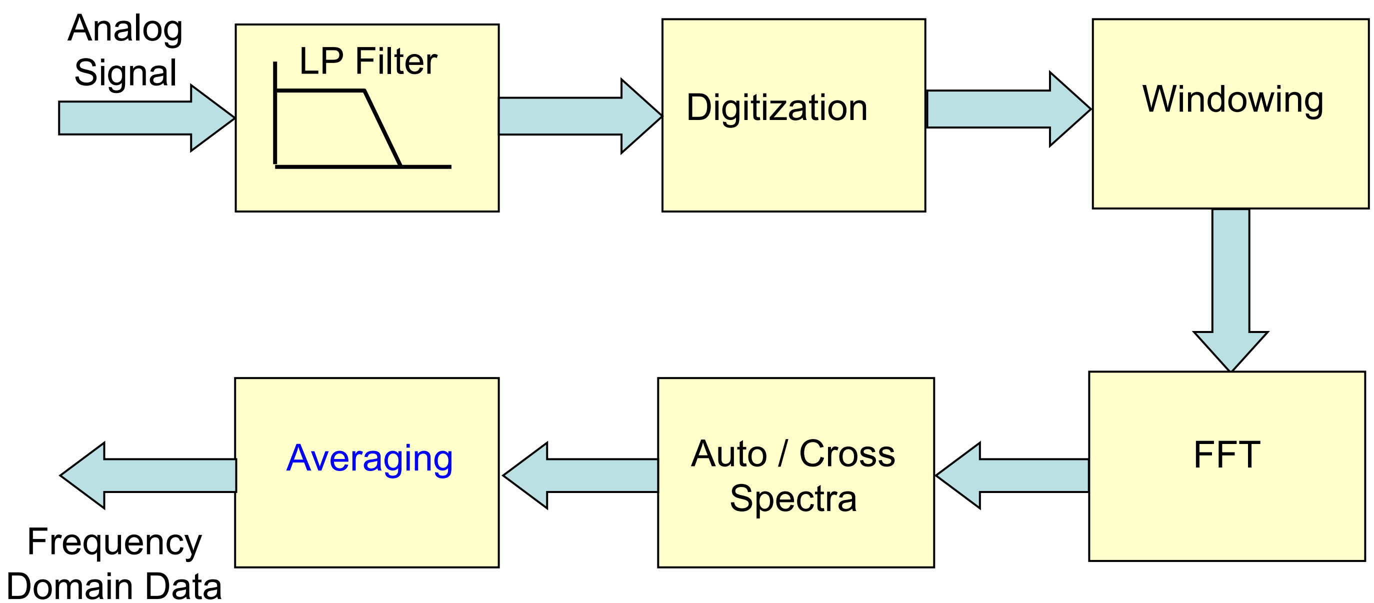


1.5 dB amplitude error



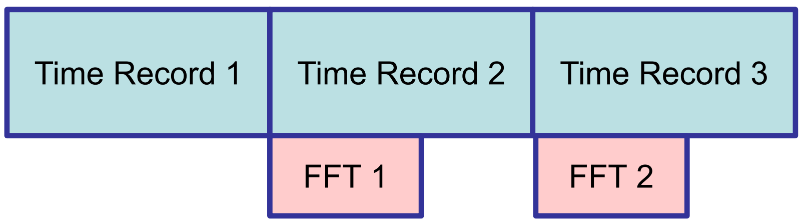
0.1 dB amplitude error

Signal Processing

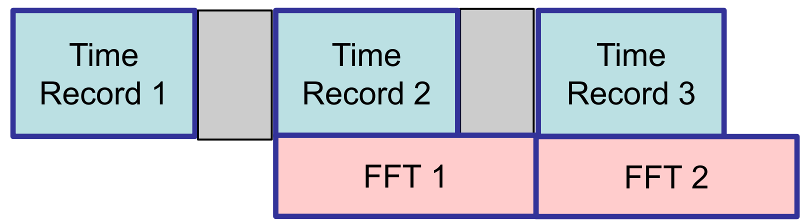


Averaging Data Processing Speed

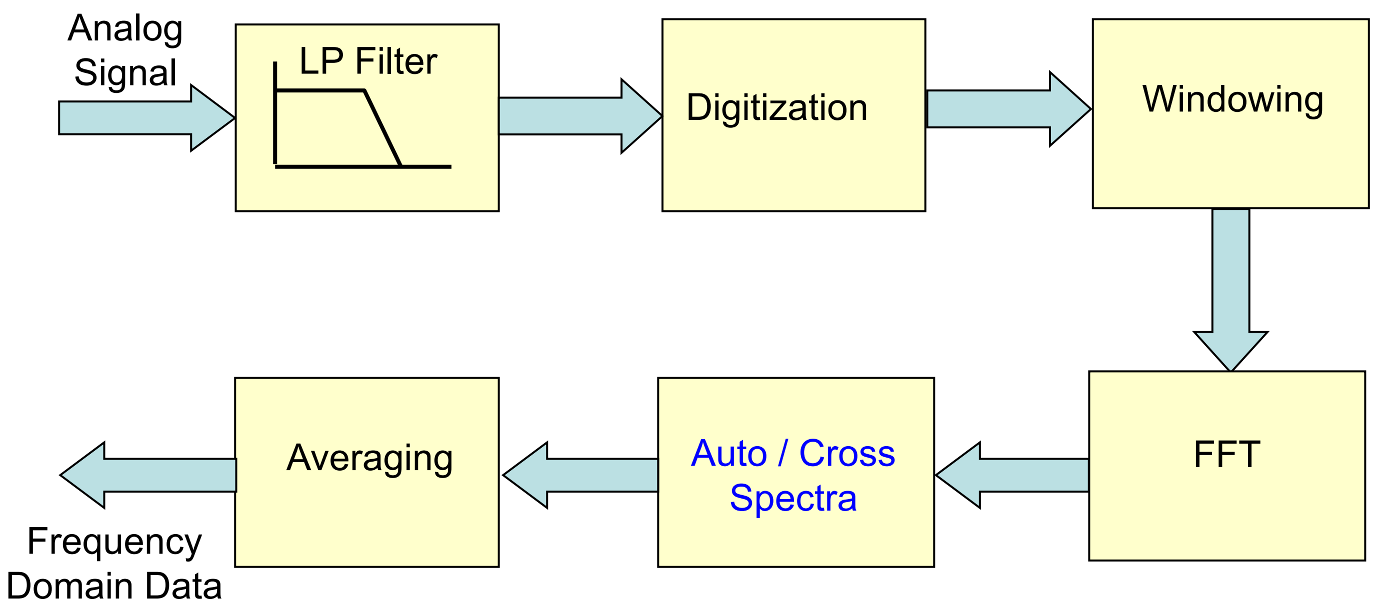
Real Time Operation



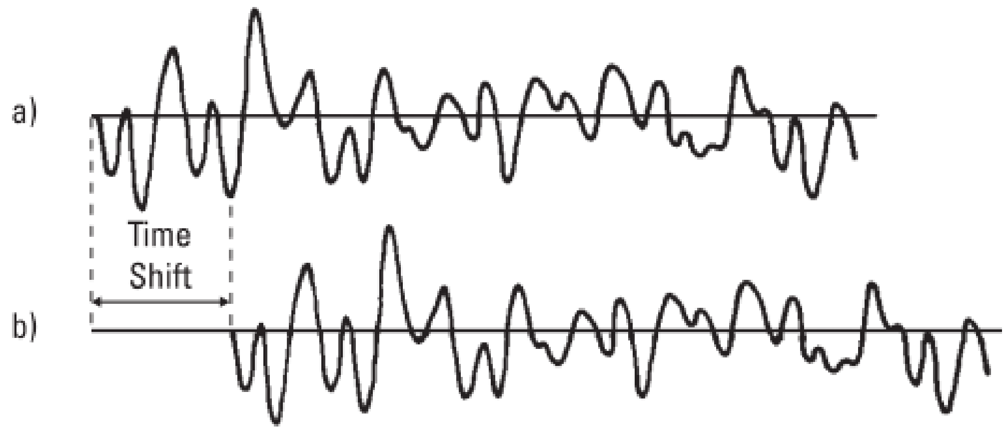
Non-Real Time Operation



Signal Processing



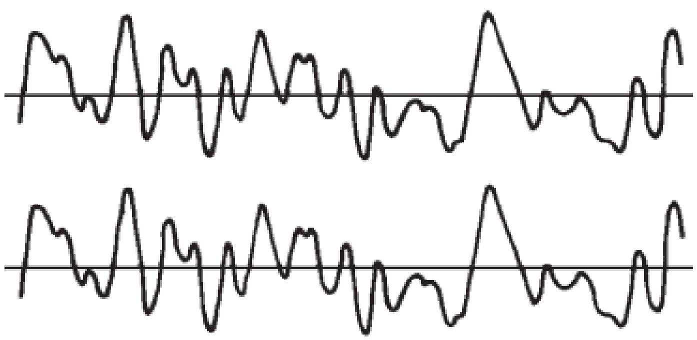
Auto Correlation



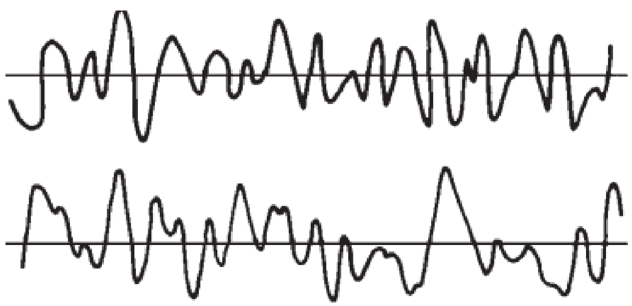
- There is a problem in averaging time displaced signals
- We can plot correlation as a function of the time shift

Auto / Cross Correlation

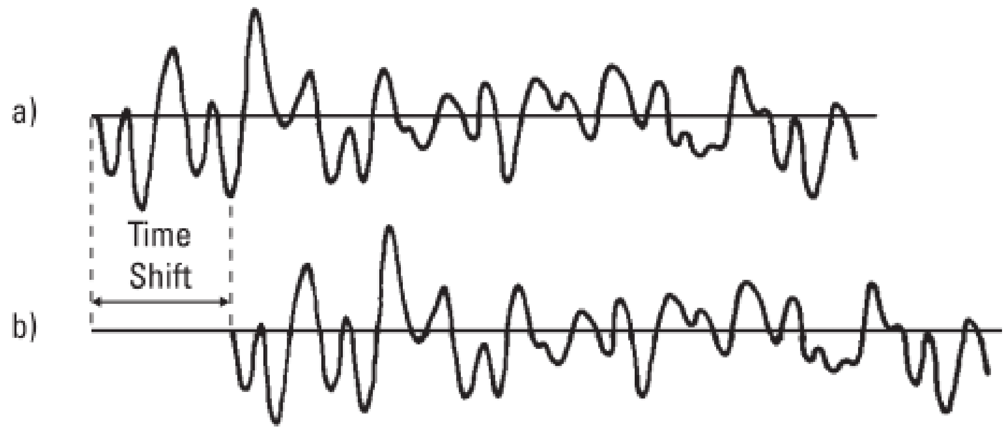
Identical Signals (Autocorrelation)



Different Signals (Cross Correlation)



Auto Correlation

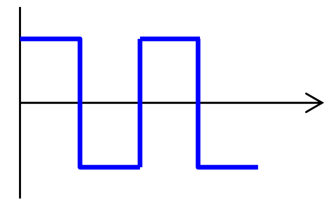


$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$

$\tau =$ time shift

Auto Correlation Square Wave

$\tau = 0$



1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0

×

1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0

1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

$= \frac{16}{N} = 1.0$

$\tau = \Delta t$

1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0

×

1 step →

-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0

-1.0	1.0	1.0	1.0	-1.0	1.0	1.0	1.0	-1.0	1.0	1.0	1.0	-1.0	1.0	1.0	1.0

$= \frac{8}{N} = 0.5$

$N = 16$ samples

Auto Correlation Square Wave

$\tau = 4\Delta t$

	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
									×							
4 steps →	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0
	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0	-1.0

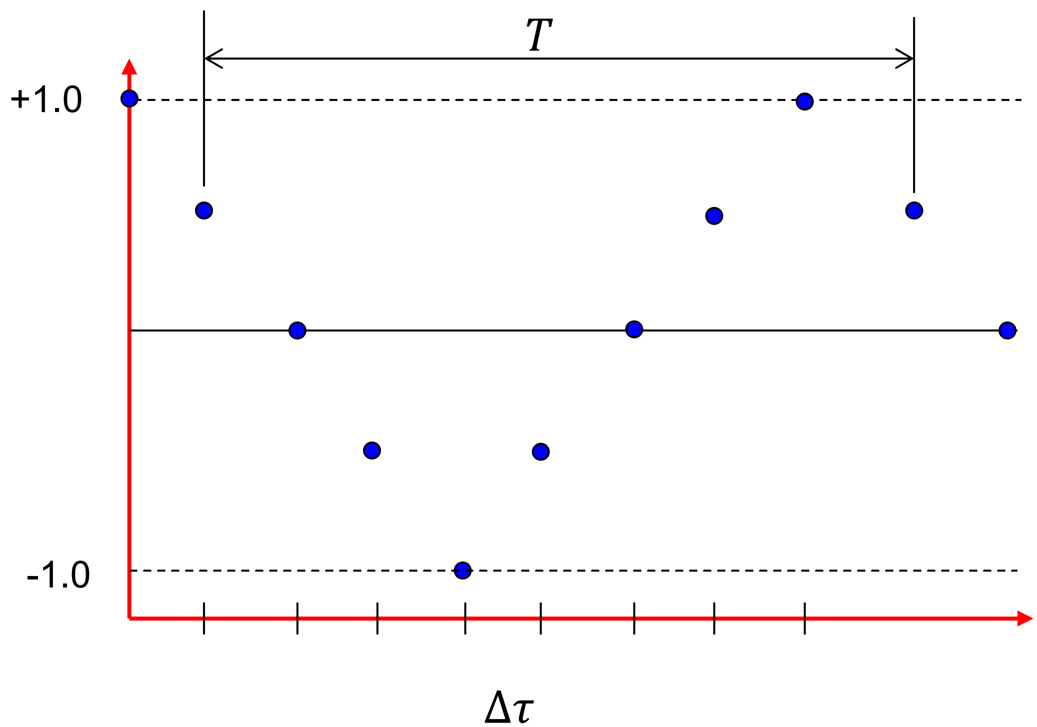
$$= -\frac{16}{N} = -1.0$$

$\tau = 8\Delta t$

	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
									×							
8 steps →	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

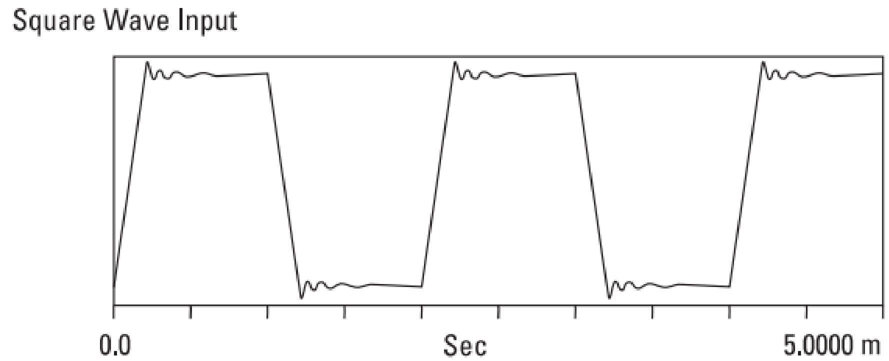
$$= \frac{16}{N} = 1.0$$

Auto Correlation Square Wave

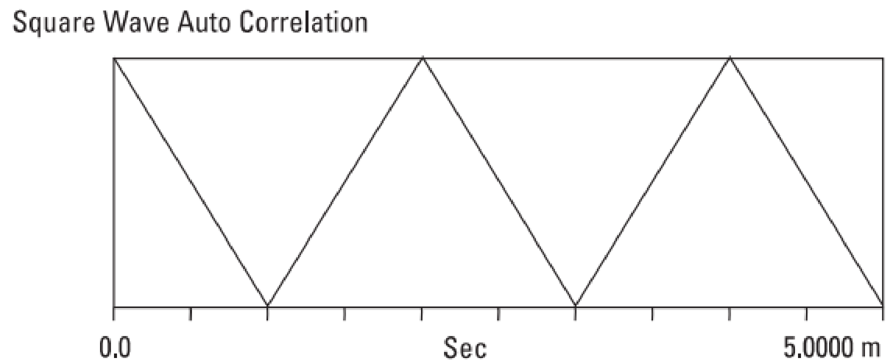


Auto Correlation

$$y(t)=\frac{4A}{\pi}\left(\begin{array}{c}\sin(\omega t)+\frac{1}{3}\sin(3\omega t)+\frac{1}{5}\sin(5\omega t)+\\ \cdots+\frac{1}{n}\sin(n\omega t)+\cdots\end{array}\right)$$



$$y(t)=\frac{8A^2}{\pi^2}\left(\begin{array}{c}\cos(\omega t)-\frac{1}{9}\cos(3\omega t)+\frac{1}{25}\cos(5\omega t)-\\ \cdots+\frac{1}{n^2}\cos(n\omega t)+\cdots\end{array}\right)$$



Auto Correlation

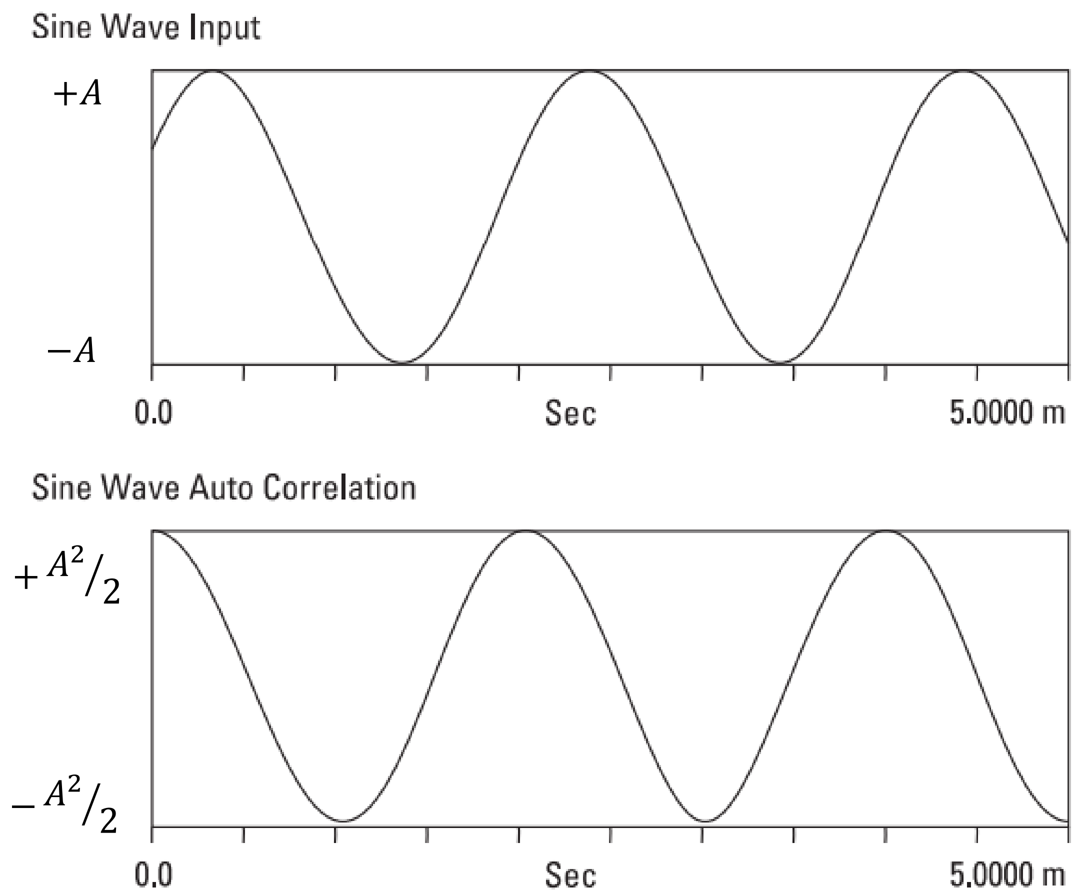
$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)x(t+\tau)dt$$

Example $x(t) = A \cdot \sin(\omega t + \varphi)$

Autocorrelation of a sine wave

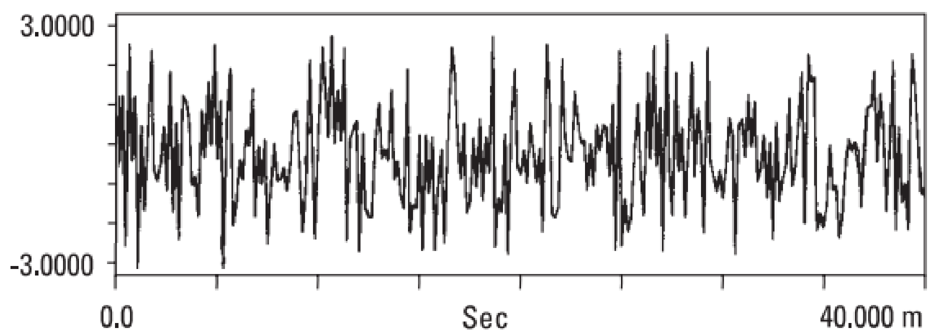
$$R_{xx}(\tau) = \frac{A^2}{2}\cos(\omega t)$$

Auto Correlation

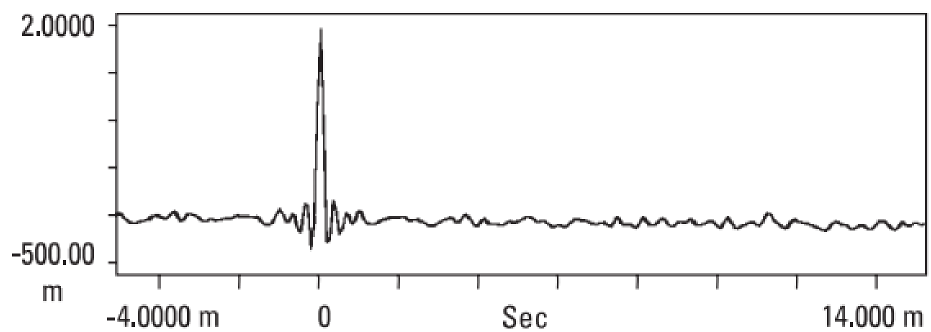


Auto Correlation

a) Time record of random noise

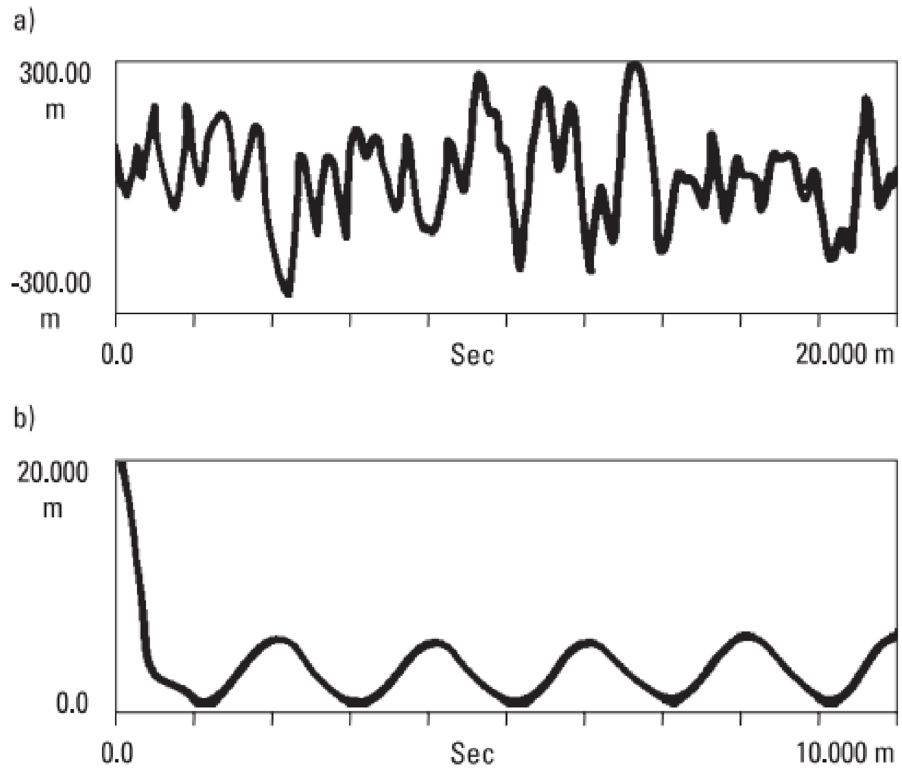


b) Auto correlation of random noise



Auto Correlation

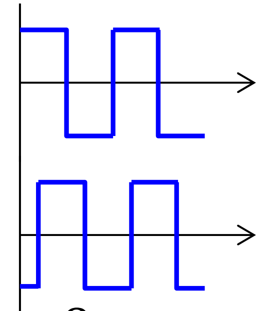
Autocorrelation of sine wave buried in noise after many averages.



Cross Correlation 2 Square Waves

$\tau = 0$

Wave x	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
×																
Wave y	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0
	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0



$$= \frac{8}{N} = 0.5$$

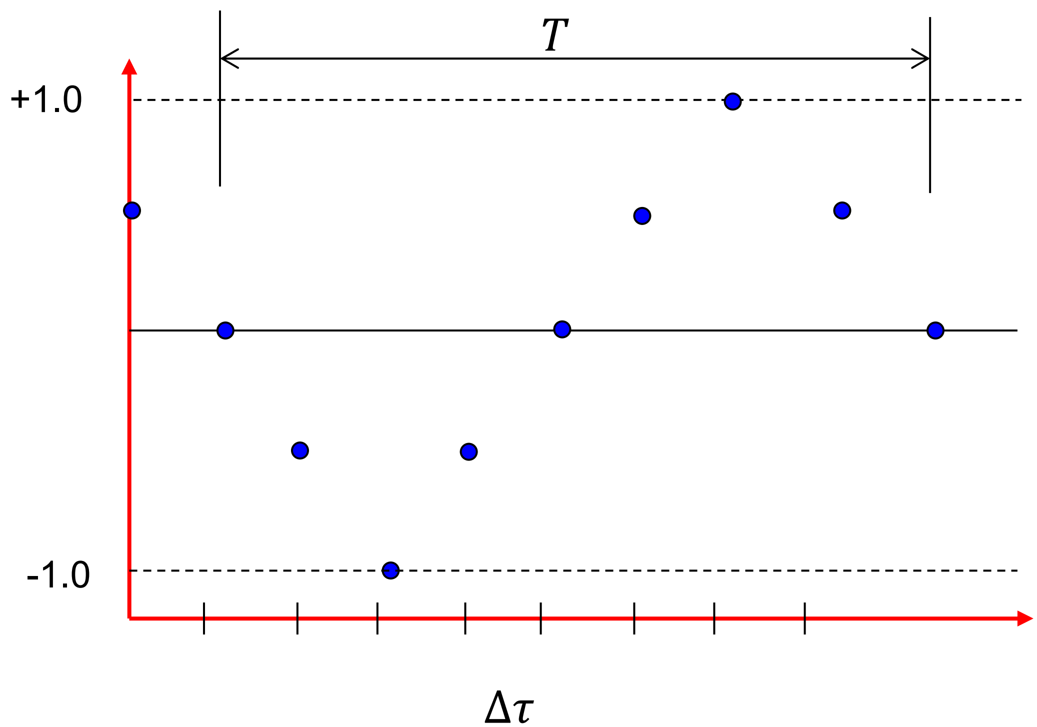
$\tau = \Delta t$

	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
×																
1 step →	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0	1.0	1.0	1.0	1.0	-1.0	-1.0
	-1.0	-1.0	1.0	1.0	-1.0	-1.0	1.0	1.0	-1.0	-1.0	1.0	1.0	-1.0	-1.0	1.0	1.0

$$= \frac{0}{N} = 0.0$$

$N = 16$ samples

Cross Correlation 2 Square Wave



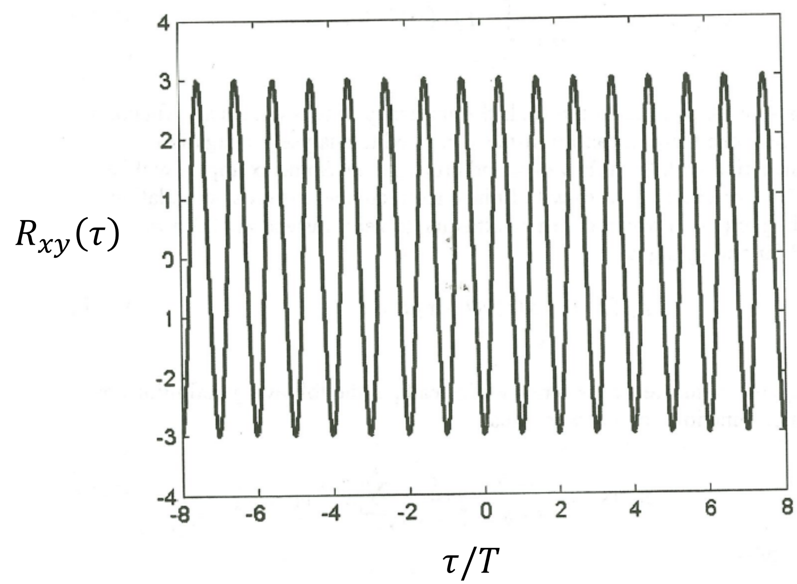
Cross Correlation

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau)dt$$

$x(t) = A_1 \sin(\omega t + \varphi)$
$y(t) = A_2 \sin(\omega t + \varphi + \vartheta)$

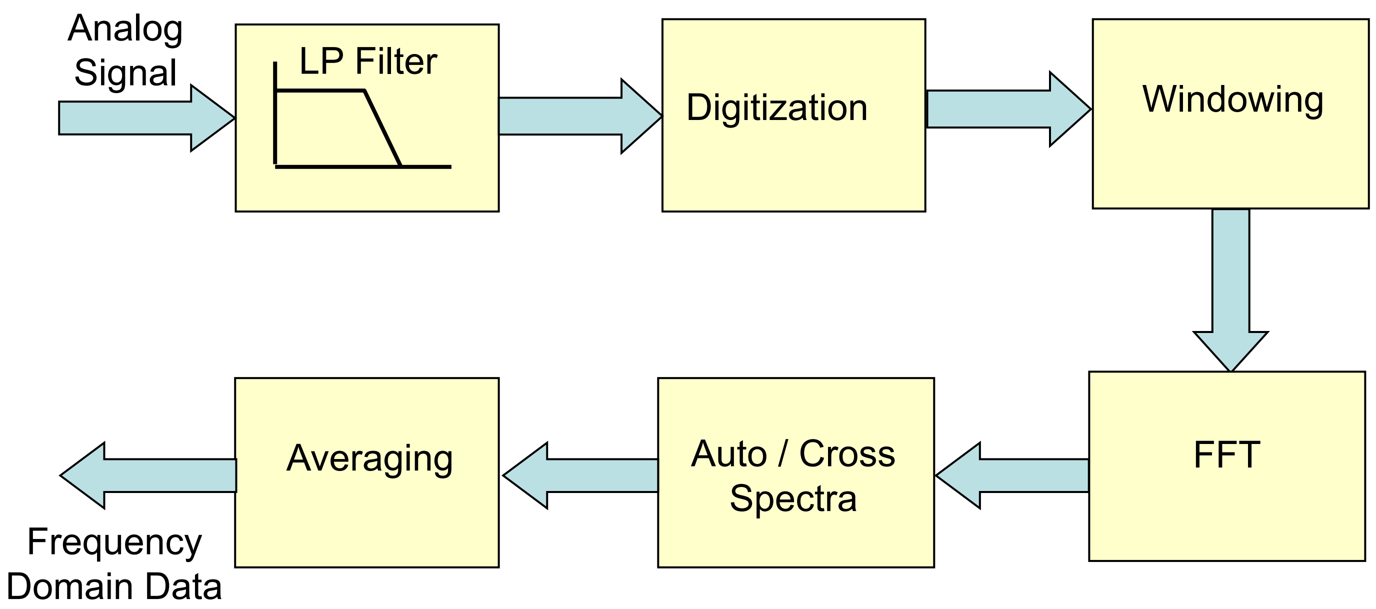
$$R_{xy}(\tau) = \frac{A_1 A_2}{2} \cos(\omega\tau + \vartheta)$$

Example
$x(t) = 2\sin(\omega t)$
$y(t) = 3\sin(\omega t + \pi)$

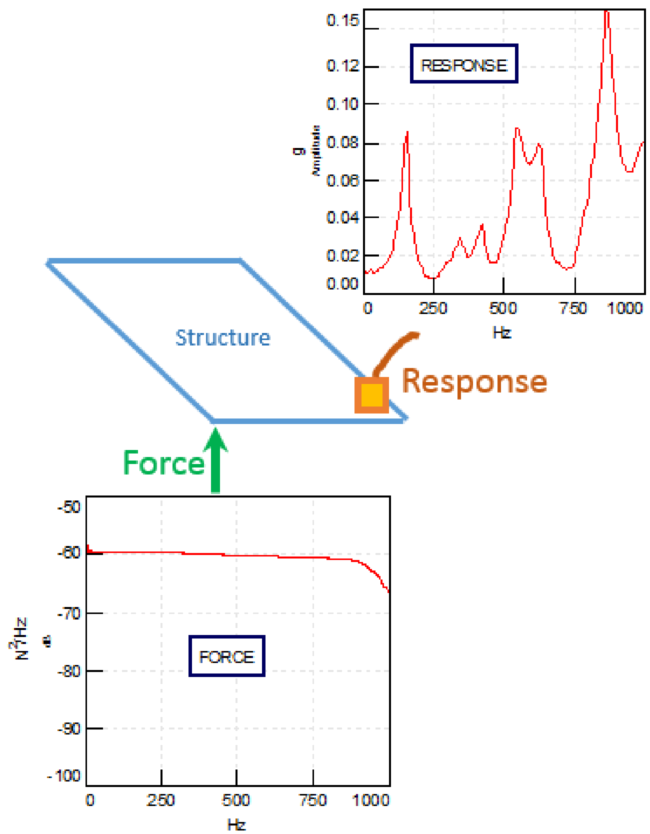


Boden et al., 2010

Signal Processing

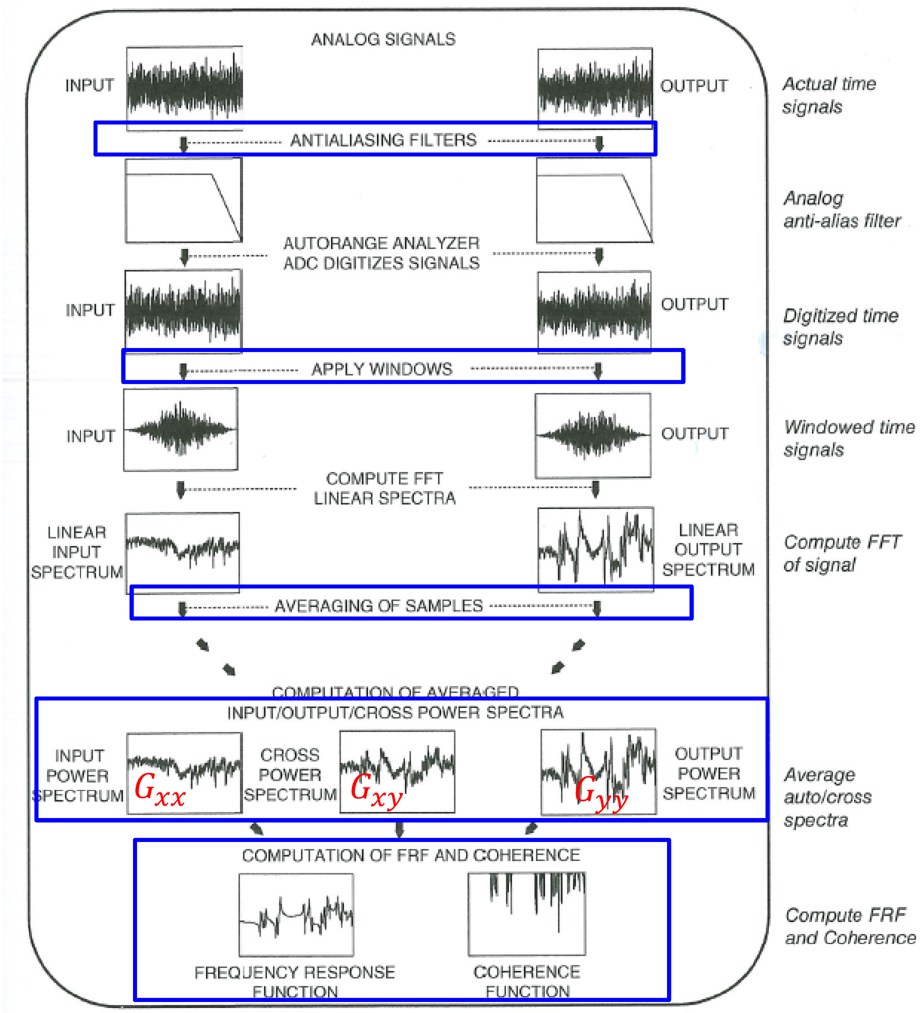


Transfer Function Measurement



Siemens, 2020

The Measurement Chain



From P. Avitabile (2017).

Transfer Function Measurement

Transfer Function Measurement

$$H_1(f) = \frac{Y(f)X'(f)}{X(f)X'(f)} = \frac{G_{xy}}{G_{xx}}$$

Use if output is more susceptible to noise

$$H_2(f) = \frac{Y(f)Y'(f)}{X(f)Y'(f)} = \frac{G_{yy}}{G_{yx}}$$

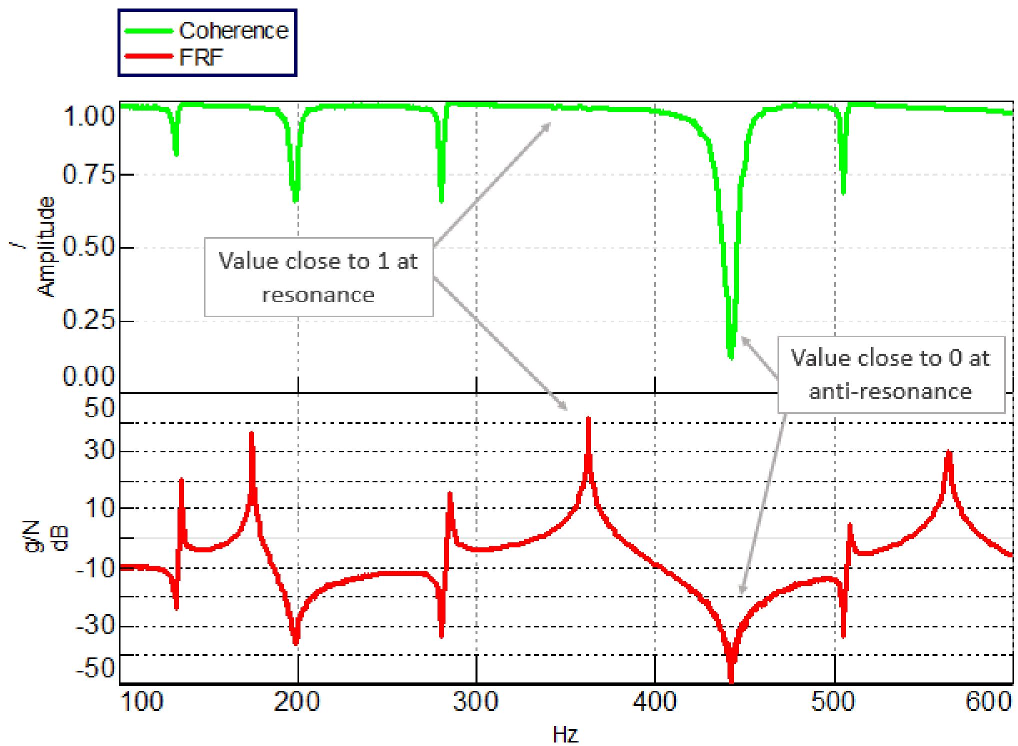
Use if input is more susceptible to noise

Transfer Function Measurement

Coherence

$$\gamma_{xy}^2(f) = \frac{G_{xy}(f)G'_{xy}(f)}{G_{xx}(f)G_{yy}(f)}$$

https://community.plm.automation.siemens.com/t5/Testing-Knowledge-Base/What-is-a-Frequency-Response-Function-FRF/ta-p/354778



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